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Article *in* Stochastics · January 1979

DOI: 10.1080/17442507908833128

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Stochastics, 1979, Vol. 2, pp. 213–225
0090-2234/79/0204-0213 $04.50/0

Optimal Sampling of Independent Increment Processes

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(*Received March 3, 1978; in final form July 24, 1978*)

A sample function drawn from a large class of independent increment processes with one of two sets of parameters is observed at evenly spaced time instants. The error probabilities associated with a standard hypothesis testing problem are studied as a function of the time interval between observations, holding the total number of observations fixed. A partial solution is obtained to the problem of choosing the optimal sampling rate.

1. INTRODUCTION

We consider a classical hypothesis testing problem of discriminating between two continuous-time real valued stochastic processes, $x(t)$ and $x'(t)$, with known parameters, by observing a sample function over a finite time interval $[0,T]$. It is well known [7], [10], that for certain (quite general) cost criteria, computing a (likelihood) functional from the observed sample function and comparing the result with a threshold is the optimal test, and that the performance of this test can be measured by various error probabilities (see the Appendix for a tutorial discussion as well as [4], [8], [9], [14], [15], [16], [17], [19]).

If the observed sample function is monitored *continuously* in time, then an important mathematical question is whether the probability distributions, ρ and ρ', of the two processes, are mutually orthogonal (on $[0,T]$), since in that case the probability of error can be theoretically zero (assuming ideal measuring apparatus). Indeed, most of the relevant mathematical statistics literature has concentrated on finding conditions which imply mutual orthogonality, e.g. [2], [3], [7], [17], [19], and on estimating error probabilities only in the case of nonsingularity (see [19],

†Research supported in part by NSF Grant MCS77-20683.

[15], [12], [13], [14]). The problem of estimating error probabilities when the data is observed *discretely* in time (for *continuous* time processes) has apparently not been treated previously, although results in [16] are closely related, and yet it is of great interest in many applications where the sampling rate and the total number of observations are parameters that must be chosen by the experimenter in some reasonably efficient way. The relevance of this problem is most apparent in those cases where ρ and ρ' are mutually orthogonal (for continuous time), even though the discrete time error probabilities are nonzero.

From this point on we suppose that our observed random process is sampled at N equally spaced time instants, $t=\Delta, 2\Delta, 3\Delta, \ldots, N\Delta$, and consider the problem of determining, for fixed N, the optimal choice of Δ to minimize some particular error probability. This *optimal sampling problem* is clearly a difficult one to treat in complete generality; here we are concerned with a *qualitative* solution of the problem, and only for a specific class of processes.

Our solution is a qualitative one in that we investigate only the limiting error probabilities as $\Delta \to 0$ (respectively, $\Delta \to \infty$). These limiting values are important for two reasons. First, they will eventually be needed in a determination of the optimal value of Δ. Second, they give important partial information to the experimenter; for example, if the limiting error probability is zero as $\Delta \to 0$ (respectively, $\Delta \to \infty$), the experimenter is justified in choosing Δ as small (respectively, as large) as possible, while limiting values of one as $\Delta \to 0, \infty$ suggest some intermediate choice of Δ should be made. Our presentation is qualitative in a further way, since we actually do not evaluate the error probabilities themselves, but rather (for technical convenience and ease of exposition) the *Kakutani product* (a generalized Hellinger integral, see, e.g. Brody (1971)) between the two discrete time probability measures. The Kakutani product is defined below, but for our purposes here it suffices to note that it is often an adequate estimate of various error probabilities (see the Appendix as well as [4], [8], [9], [14], [19]) and moreover, that if it tends to zero or one (as $\Delta \to 0$, or as $\Delta \to \infty$), then so do the error probabilities.

The specific processes we deal with here are real-valued time-homogeneous independent increment processes. With the further assumption of right continuity, and that $x(0)=x'(0)=0$ a.s., such a process is determined by its characteristic functional

$$E(e^{i\omega(x(t)-x(s))}) = \exp\left[(t-s)\left(i\omega a - \tfrac{1}{2}S\omega^2 + \int\left\{e^{i\omega u} - 1 - \frac{i\omega u}{1+u^2}\right\} dM(u)\right)\right] \tag{1.1}$$

for $0 \leq s < t \leq T$, where

$$-\infty < a < +\infty, \; S \geq 0, \text{ and } \int_{u \neq 0} \frac{u^2}{1+u^2}\, dM(u) < \infty$$

There is a similar formula for $x'(t)$ with corresponding parameters (a', S', M').

Recall that if μ, μ' are arbitrary σ-finite measures with a product measure on a measure space (Ω, A), then the *Kakutani product*, with parameter $r \in (0,1)$, between μ and μ' is denoted $h_r(\mu, \mu')$,

$$h_r(\mu,\mu') = \int (d\mu/d\nu)^r (d\mu'/d\nu)^{1-r}\, d\nu \tag{1.2}$$

where $\mu, \mu' \ll \nu$ [the definition is independent of the choice of ν, and many such ν exist; a class of examples is $\nu = c(\mu+\mu')$ where c is a positive real constant]; $r=\frac{1}{2}$ is the classical Hellinger integral. From Hölder's inequality it is straightforward to show $0 \leq h_r(\mu,\mu') \leq 1$, with $h_r = 1$ iff $\mu = \mu'$, and $h_r = 0$ iff $\mu \perp \mu'$. In our optimal sampling problem, we consider $h_r(\mu,\mu')$ with μ the joint distribution on $\mathbf{R}^N$ of $\{x(\Delta), x(2\Delta), x(3\Delta), \ldots, x(N\Delta)\}$, and with μ' similarly related to x'. It is easily seen that for independent increment processes, we have

$$h_r(\mu,\mu') = [h_r(\rho_\Delta, \rho'_\Delta)]^N$$

where ρ_Δ (respectively ρ'_Δ) is the distribution on $\mathbf{R}$ of $x(\Delta)$ (respectively $(x'(\Delta))$.

After bounding the actual error probabilities by the Kakutani product, we see that with N fixed, our task is to choose Δ so as to minimize $G(\Delta) \equiv h_r(\rho_\Delta, \rho'_\Delta)$. As explained above, our main focus here is to evaluate (when they exist)

$$G(0) = \lim_{\Delta \to 0} G(\Delta) \quad \text{and} \quad G(\infty) = \lim_{\Delta \to \infty} G(\Delta).$$

The existence and evaluation of these limits is intimately connected with the notion of *domains of attraction* which involves generalizations of the central limit theorem. In the case $\Delta \to \infty$, this is a well-studied area, with extensive treatments in Gnedenko and Kolmogorov [6] and Feller [5]. The case $\Delta \to 0$ has apparently not been previously studied. The idea is that under appropriate regularity assumptions there exist (unique) constants $\underline{\alpha}$ and $\bar{\alpha}$ so that for some appropriate choice of centering constants $\underline{\Theta}(\Delta)$ and $\bar{\Theta}(\Delta)$, the following limits exist and have nontrivial distributions:

$$\underline{x} = \lim_{\Delta \to 0} x(\Delta)/\Delta^{1/\underline{\alpha}} - \underline{\Theta}(\Delta), \tag{1.3a}$$

$$\bar{x} = \lim_{\Delta \to \infty} x(\Delta)/\Delta^{1/\bar{\alpha}} - \bar{\Theta}(\Delta) \tag{1.3b}$$

$\underline{x}$ and $\bar{x}$ are necessarily stable random variables with exponents $\underline{\alpha}$ and $\bar{\alpha}$ (which we shall define below), and one expects intuitively that $G(0)=0$ if $\underline{\alpha}$ and the corresponding $\underline{\alpha}'$ differ, or if $\underline{\alpha}=\underline{\alpha}'$ but $\underline{\Theta}(\Delta)-\underline{\Theta}'(\Delta)\to\pm\infty$. In other words, two processes with sufficiently different scaling/centering parameters can be easily discriminated between. On the other hand, if $\underline{\alpha}=\underline{\alpha}'$ and $\underline{\theta}(\Delta)-\underline{\theta}'(\Delta)\to\delta$, for some finite δ, then we expect $G(0)$ to equal the Kakutani product between the distributions of $\underline{x}+\delta$ and $\underline{x}'$. Analogous statements apply to $G(\infty)$.

In this section, limits of random variables will always denote weak of $G(0)$ and $G(\infty)$. Since it is not our primary purpose here to give a complete analysis of domains of attraction, our results (with the exception of Theorem 1 below which treats the special case $\alpha=2$), will not be stated for general independent increment processes. Instead, we will consider (in Theorems 2 and 3) a restricted class of processes (finite sums of independent stable processes), which is sufficiently rich to exhibit most phenomena one might encounter in general. These theorems and their proofs can be easily extended to processes satisfying (1.3). This results in the statements of the theorems being quite long compared to the proofs, because of a great many special cases that must be considered in a thorough treatment. Section two also includes tables intended to provide the experimenter with useful partial information as to how to best choose a sampling interval Δ given the parameters of x and x'. In Section 3 we give the proofs of all our results.

2. THE MAIN RESULTS

In this section, limits of random variables will always denote weak limits, i.e. $x_n\to x$ whenever $E(\exp(i\omega x_n))\to E(\exp(i\omega x))$, for all real ω. We will also use the notation $h_r(Z,Z')$ to denote the Kakutani product between the distributions of the random variables Z,Z'.

THEOREM 1 *If the characteristic functional of $x(t)$ is as given by* (1.1), *then $x(\Delta)/\Delta^{1/2}\to Z_S$ as $\Delta\to 0$, where Z_S is a Gaussian random variable with mean zero and variance S ($Z_S=0$ a.s. when $S=0$). It follows that if either S or S' is nonzero, then $G(0)=h_r(Z_S,Z_{S'})$; in particular, we have that*

$$G(0)=\begin{cases}0 & \text{if } S=0\neq S'\\ 0 & \text{if } S\neq 0=S'\\ \{S^{(1-r)}S'^{r}/[(1-r)S+rS']\}^{1/2}<1 & \text{if } S\neq S';\ S,S'>0\\ 1 & \text{if } S=S'>0\end{cases}\tag{2.1}$$

Given any $\alpha\in(0,2]$, $\beta\in[-1,1]$, and $\gamma>0$, we consider $Y(\alpha,\beta,\gamma)$, the

(possibly asymmetric) stable random variable of exponent α, which satisfies

$$E[\exp(i\omega Y(\alpha,\beta,\gamma))]=\exp[\Psi(\omega;\alpha,\beta,\gamma)]$$

where

$$\Psi(\omega;\alpha,\beta,\gamma)=\begin{cases}-\gamma|\omega|^{\alpha}[1+i\beta\tan(\pi\alpha/2)\mathrm{sign}(\omega)], & \alpha\neq\\ -\gamma|\omega|[1+i\beta\ln|\omega|\mathrm{sign}(\omega)], & \alpha=1.\end{cases}\tag{2.2}$$

When $\alpha=2$, $Y(2,\beta,\gamma)$ is a Gaussian random variable with mean zero and variance 2γ, and is independent of β. Each $Y(\alpha,\beta,\gamma)$ has an absolutely continuous distribution on the real line with strictly positive density unless $\alpha<1$ and $\beta=\pm1$ in which case the density vanishes on either the left (for $\beta=+1$) or the right (for $\beta=-1$) half line and is strictly positive on the other half line [see Lukacs (1970)].

To simplify the statement of the next theorem, we note that the above properties imply for $\gamma,\gamma'>0$ and $-\infty<\delta,\delta'<\infty$,

$$h_r(Y(\alpha,\beta,\gamma),Y(\alpha,\beta',\gamma'))=\begin{cases}0 & \text{if } \alpha<1, \beta=-\beta'=\pm1\\ 1 & \text{if } \beta=\beta', \gamma=\gamma' (\text{or } \alpha=2 \text{ and } \gamma=\gamma\\ \zeta\in(0,1) & \text{otherwise}\end{cases}\tag{2.3}$$

$$h_r(Y(1,\beta,\gamma)+\delta,Y(1,\beta',\gamma')+\delta')=\begin{cases}1 & \text{if } \beta=\beta',\gamma=\gamma',\delta=\delta'\\ \zeta\in(0,1) & \text{otherwise}\end{cases}\tag{2.4}$$

In the next two theorems, we assume that $x(t)$ and $x'(t)$ are finite sums of independent stable processes and a pure drift process so that (1.1) is replaced by

$$E\{\exp[i\omega(x(t)-x(s))]\}=\exp\left\{(t-s)\left[i\delta\omega+\sum_{j=1}^{N}\Psi(\omega;\alpha_j,\beta_j,\gamma_j)\right]\right\}\tag{2.5}$$

with

$$-\infty<\delta<\infty,\ 2\geqq\alpha_1>\alpha_2>\ldots>\alpha_N>0,\ \beta_j\in[-1,1], \text{ and } \gamma_j>0 \text{ for all } j;$$

a similar formula applies to $x'(t)$ with parameters δ', α_j', β_j', γ_j', and N'. If either $N=0$ or $N'=0$, then $G(\Delta)\equiv0$ for all Δ unless $N=N'=0$ and $\delta=\delta'=0$, in which case $G(\Delta)=1$ for all Δ; thus, we will assume from this point on that $N,N'\geqq1$. We then define $(\underline{\alpha},\underline{\beta},\underline{\gamma})=(\alpha_1,\beta_1,\gamma_1)$ and $(\bar{\alpha},\bar{\beta},\bar{\gamma})=(\alpha_N,\beta_N,\gamma_N)$ and similarly for $x'(t)$. We also note that now $G(\Delta)<1$ for all Δ unless all the parameters of $x'(t)$ are identical to those of $x(t)$ and that $G(\Delta)>0$ for all Δ unless $\underline{\alpha},\underline{\alpha}'<1$ and also $\beta_j=-\beta_j'=1$ for all j, while $\delta-\delta'\geqq 0$ (or else $\beta_j=-\beta_j'=-1$ for all j while $\delta-\delta'\leqq 0$).

THEOREM 2. *If $x(t)$ satisfies* (2.5), *then as* $\Delta\to 0$,

$$x(\Delta)/\Delta^{1/\underline{\alpha}}-\underline{\Theta}(\Delta)\to x\equiv Y(\underline{\alpha},\underline{\beta},\underline{\gamma}), \tag{2.6}$$

where

$$\underline{\Theta}(\Delta)=\begin{cases}\delta\Delta^{1-1/\underline{\alpha}} & \text{for } \underline{\alpha}\neq 1\\ \delta+\underline{\gamma}\underline{\beta}\ln(\Delta) & \text{for } \underline{\alpha}=1.\end{cases} \tag{2.7}$$

If $\underline{\alpha}=\underline{\alpha}'$, we define

$$\underline{\delta}=\lim_{\Delta\to 0}[\underline{\Theta}(\Delta)-\underline{\Theta}'(\Delta)]=\begin{cases}0 & \text{if } \underline{\alpha}=\underline{\alpha}'>1\\ 0 & \text{if } \underline{\alpha}=\underline{\alpha}'<1 \text{ and } \delta=\delta'\\ \delta-\delta' & \text{if } \underline{\alpha}=\underline{\alpha}'=1 \text{ and } \underline{\gamma}\underline{\beta}=\underline{\gamma}'\underline{\beta}'\\ \pm\infty, & \text{otherwise.}\end{cases} \tag{2.8}$$

Then

$$G(0)=\begin{cases}0 & \text{if } \underline{\alpha}\neq\underline{\alpha}'\\ 0 & \text{if } \underline{\alpha}=\underline{\alpha}', \quad \text{but } \underline{\delta}=\pm\infty.\\ h_r(\underline{x}+\underline{\delta},\underline{x}') & \text{otherwise.}\end{cases} \tag{2.9}$$

Thus, $G(0)$ is either zero, one, or in the open interval (0,1), according to (2.3) and (2.4).

THEOREM 3. *If $x(t)$ satisfies* (2.5), *then as* $\Delta\to\infty$,

$$x(\Delta)/\Delta^{1/\bar{\alpha}}-\bar{\Theta}(\Delta)\to\bar{x}\equiv Y(\bar{\alpha},\bar{\beta},\bar{\gamma}), \tag{2.10}$$

where

$$\bar{\Theta}(\Delta)=\begin{cases}\delta\Delta^{1-1/\bar{\alpha}} & \text{for } \bar{\alpha}\neq 1\\ \delta+\bar{\gamma}\bar{\beta}\ln(\Delta) & \text{for } \bar{\alpha}=1\end{cases} \tag{2.11}$$

If $\bar{\alpha}=\bar{\alpha}'$, we define

$$\bar{\delta}=\lim_{\Delta\to\infty}[\bar{\Theta}(\Delta)-\bar{\Theta}'(\Delta)]=\begin{cases}0 & \text{if } \bar{\alpha}=\bar{\alpha}'<1\\ 0 & \text{if } \bar{\alpha}=\bar{\alpha}'>1 \text{ and } \delta=\delta'\\ \delta-\delta' & \text{if } \bar{\alpha}=\bar{\alpha}'=1 \text{ and } \bar{\gamma}\bar{\beta}=\bar{\gamma}'\bar{\beta}'\\ \pm\infty & \text{otherwise.}\end{cases} \tag{2.12}$$

Then

$$G(\infty)=\begin{cases}0 & \text{if } \bar{\alpha}\neq\bar{\alpha}'\\ 0 & \text{if } \bar{\alpha}=\bar{\alpha}' \text{ but } \bar{\delta}=\pm\infty\\ h_r(\bar{x}+\bar{\delta},\bar{x}') & \text{otherwise.}\end{cases} \tag{2.13}$$

In order to unravel the conditions of Theorems 2 and 3, we present tables below which tell when $G(0)$ (Table 1) and $G(\infty)$ (Table 2) are either zero, one, or in the open interval (0,1). The nine possibilities are arranged graphically in the final table (Table 3), where we suggest how to choose Δ: either as small as possible (denoted **S**), as large as possible (denoted **L**), or at some intermediate value (denoted *I*). Note that no suggestion can be made when both $G(0)$ and $G(\infty)$ are in the open interval (0,1).

TABLE I

Parameters	
$\underline{\alpha}\neq\underline{\alpha}'$	0
$\underline{\alpha}=\underline{\alpha}'<1,\ \delta\neq\delta'$	0
$\underline{\alpha}=\underline{\alpha}'<1,\ \delta=\delta',\ \underline{\beta}=-\underline{\beta}'=\pm 1$	0
$\underline{\alpha}=\underline{\alpha}'=1,\ \underline{\beta}\underline{\gamma}\neq\underline{\beta}'\underline{\gamma}'$	0
$\underline{\alpha}=\underline{\alpha}'<1,\ \underline{\beta}=\underline{\beta}',\ \underline{\gamma}=\underline{\gamma}',\ \delta=\delta'$	1
$\underline{\alpha}=\underline{\alpha}'>1,\ \underline{\beta}=\underline{\beta}',\ \underline{\gamma}=\underline{\gamma}'$	1
$\underline{\alpha}=\underline{\alpha}'=2,\ \underline{\gamma}=\underline{\gamma}'$	1
Otherwise	$\zeta\in(0,1)$

TABLE II

$G(\infty)$	
Parameters	$G(\infty)$
$\bar{\alpha}\neq\bar{\alpha}'$	0
$\bar{\alpha}=\bar{\alpha}'>1,\ \delta\neq\delta'$	0
$\bar{\alpha}=\bar{\alpha}'<1,\ \bar{\beta}=-\bar{\beta}'=\pm 1$	0
$\bar{\alpha}=\bar{\alpha}'=1,\ \bar{\beta}\bar{\gamma}\neq\bar{\beta}'\bar{\gamma}'$	0
$\bar{\alpha}=\bar{\alpha}'<1,\ \bar{\beta}=\bar{\beta}',\ \bar{\gamma}=\bar{\gamma}'$	1
$\bar{\alpha}=\bar{\alpha}'\geqq 1,\ \bar{\beta}=\bar{\beta}',\ \bar{\gamma}=\bar{\gamma}',\ \delta=\delta'$	1
$\bar{\alpha}=\bar{\alpha}'=2,\ \bar{\gamma}=\bar{\gamma}',\ \delta=\delta'$	1
Otherwise	$\zeta\in(0,1)$

TABLE III
Qualitative results concerning optimal choice of Δ

	$G(0)=0$	$G(0)\in(0,1)$	$G(0)=1$
$G(\infty)=0$	S,L	L	L
$G(\infty)\in(0,1)$	S	—	I,L
$G(\infty)=1$	S	S,I	I

3. PROOF OF MAIN RESULTS

The main technical tools we need are developed in the following lemmas, in which we write $x_n \Rightarrow x$ to mean that $x_n \to x$ (weak convergence), while $|E\{\exp(i\omega x_n)\}|$ is bounded by some fixed L_1 function of ω, independent of n. In the proofs of these lemmas, we will use the equality $h_r(ax+b, ax'+b) = h_r(x,x')$ for real $a \neq 0$, b, which follows by a change of variables in the

LEMMA 1. *If $x_n \to x$ and $x'_n \to x'$, then*

$$\limsup_{n\to\infty} h_r(x_n, x'_n) \leq h_r(x,x'); \tag{3.1}$$

if $x_n \Rightarrow x$ and $x'_n \Rightarrow x'$, then

$$\lim_{n\to\infty} h_r(x_n, x'_n) = h_r(x,x'). \tag{3.2}$$

Proof. (3.1) is a standard result concerning Kakutani products; see for example [1], [13]. To derive (3.2), we note that $E(\exp(i\omega x_n)) \to E(\exp(i\omega x))$ in L_1 (by the Lebesgue dominated convergence theorem) so that by taking inverse Fourier transforms, we see that x_n has an absolutely continuous distribution with density denoted by f_n which is continuous and converges uniformly to the density f of x; a similar result applies to x'_n. It is then clear from the definition of h_r that (3.2) follows. Q.E.D.

LEMMA 2 *If $x_n \Rightarrow x$ and $x'_n \Rightarrow x'$ and $-\infty < \delta_n, \delta'_n < \infty$, with*

$$\lim_{n\to\infty} (\delta_n - \delta'_n) \equiv \hat{\delta}$$

then

$$\lim_{n\to\infty} h_r(x_n + \delta_n, x'_n + \delta'_n) = \begin{cases} h_r(x+\hat{\delta}, x') & \text{if } \hat{\delta} \text{ exists } \textit{and} \text{ is finite} \\ 0 & \text{if } \hat{\delta} = \pm\infty. \end{cases}$$

Proof. First, suppose $\hat{\delta}$ exists and is finite. Then $x_n + \delta_n - \delta'_n \Rightarrow x + \hat{\delta}$, since $|E\{\exp[i\omega(x_n+\delta)]\}|$ has the same L_1 bound as $|E\{\exp[i\omega x_n]\}|$. The desired conclusion now follows from Lemma 1 and the fact that $h_r(x_n+\delta_n, x'_n+\delta'_n) = h_r(x_n + (\delta_n - \delta'_n), x'_n)$. Now suppose $\hat{\delta} = \pm\infty$. Since the density of x tends to zero at $\pm\infty$ (by the Riemann–Lebesgue lemma), it follows that on any bounded interval the density of $x_n + (\delta_n - \delta'_n)$ must tend uniformly as $n \to \infty$ to zero. A standard argument yields the desired conclusion. Q.E.D.

LEMMA 3 *If $x_n \Rightarrow x$ and $x'_n \to 0$, then for any δ'_n, $-\infty < \delta'_n < \infty$,*

$$\lim_{n\to\infty} h_r(x_n, x'_n + \delta'_n) = ($$

Proof By considering subsequences, we may assume that either $\delta'_n \to \pm\infty$ or $\delta_n \to \delta'$. If $\delta'_n \to \delta'$, then the result follows by (3.1) and the proof of Lemma 1. If $\delta'_n \to \pm\infty$, then the result follows by the equality $h_r(x_n, x'_n + \delta'_n) = h_r(x_n - \delta'_n, x'_n)$ together with the proof of Lemma 2 and standard arguments. Q.E.D.

Proof of Theorem 1 From Lemma 1 and the equality $h_r(x(\Delta), x'(\Delta)) = h_r(x(\Delta)/\Delta^{1/2}, x'(\Delta)/\Delta^{1/2})$, it suffices to prove that $x(\Delta)/\Delta^{1/2} \Rightarrow Z_S$ as $\Delta \to 0$ when $S > 0$, and $x(\Delta)/\Delta^{1/2} \to 0$ as $\Delta \to 0$ when $S = 0$. Since

$$|E[\exp(i\omega x(\Delta)/\Delta^{1/2})]| \leq e^{-\frac{1}{2}S\omega^2}$$

independently of Δ by (1.1), we need only show that $\{i\omega\Delta^{1/2} + \Delta F(\omega/\Delta^{1/2}) \to 0\}$ pointwise as $\Delta \to 0$, where

$$F(\omega) = \int_{u\neq 0} \left(e^{i\omega u} - 1 - \frac{i\omega u}{1+u^2} \right) dM(u). \tag{3.3}$$

We define

$$F_\varepsilon(\omega) = \int_{0 \neq |u| \leq \varepsilon} (e^{i\omega u} - 1 - i\omega u)\, dM(u)$$

and observe that since $i\omega\Delta^{1/2} \to 0$ and $\Delta[F(\omega/\Delta^{1/2}) - F_\varepsilon(\omega)] \to 0$ (as can be seen by simple estimates), it suffices to show that

$$\lim_{\varepsilon\to 0} \limsup_{\Delta\to 0} |\Delta F_\varepsilon(\omega/\Delta^{1/2})| = 0$$

This in turn follows by observing that there exists a real positive constant $C > 0$ such that

$$|e^{i\omega u} - 1 - i\omega u| \leq C(\omega u)^2$$

for all real ωu, so that

$$|F_\varepsilon(\omega)| \leq C\omega^2 \int_{0 \neq |u| \leq \varepsilon} u^2\, dM(u)$$

with

$$\int_{0 \neq |u| \leq \varepsilon} u^2\, dM(u) \to 0 \quad \text{as } \varepsilon \to 0$$

since

$$\int_{u \neq 0} \frac{u^2}{1+u^2}\, dM(u) < \infty$$

which has been shown elsewhere in a different context†

†See N. S. Landkof, *Foundations of Modern Potential Theory*, Lemma 6.10, Springer-Verlag, NY, 1972.

Proofs of Theorem 2 *and* 3 We first show that as $\Delta \to 0$,

$$x(\Delta)/\Delta^{1/\underline{\alpha}} - \underline{\Theta}(\Delta) \Rightarrow \underline{x} \tag{3.4}$$

This is a consequence of the simple estimate (obtained from (2.2)) that

$$E[\exp(i\omega x(\Delta)/\Delta^{1/\alpha})] \leqq \exp(-\gamma|\omega|^{\alpha})$$

together with the fact that

$$\Delta\Psi(\omega/\Delta^{1/\alpha};\alpha,\beta,\gamma) = \begin{array}{ll} \Psi(\omega;\alpha,\beta,\gamma) & \text{for } \underline{\alpha}=\alpha\neq 1 \\ \Psi(\omega;\alpha,\beta,\gamma)+i\beta\gamma\ln(\Delta)\omega & \text{for } \underline{\alpha}=\alpha= \\ \to 0 \quad \text{as } \Delta\to 0 \text{ for } \alpha<\underline{\alpha}. & \end{array} \tag{3.5}$$

The conclusions of Theorem 2 (for the case $\underline{\alpha}=\underline{\alpha}'$) now follow from Lemmas 1 and 2. If $\underline{\alpha}\neq\underline{\alpha}'$ (say $\underline{\alpha}'<\underline{\alpha}$), we note from (3.5) that

$$x'(\Delta)/\Delta^{1/\underline{\alpha}} - \delta'\Delta^{1-1/\underline{\alpha}} \to 0. \tag{3.6}$$

The fact that $G(0)=0$ in this case is a consequence of Lemma 3

The proof of Theorem 3 is essentially identical. Q.E.D.

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Appendix

APPLICATION OF CHERNOFF TYPE BOUNDS TO THE BOUNDING OF ERRORS IN STATISTICAL HYPOTHESIS TESTING

In this Appendix we summarize some well known properties concerning a very simple and useful technique due to H. Chernoff for bounding errors arising in statistical hypothesis testing problems. See [4], [7], [8], [9], [10], [15], [16], [19] and the references therein for more information.

Let $x_1(t),x_2(t)$ be separable versions of two different stochastic processes taking values in a common state space, where t is time, $t\in E$, E a bounded set. Time will be either discrete, $E=\{0,1,\ldots,N\}$, or continuous, $E=[0,T)$. ρ and ν denote the respective probability measures of the two processes on D, the space of right continuous paths with left hand limits everywhere defined. All probability measures are defined on the measure space (Ω,A), where Ω is the set of elementary events, and A is the σ algebra or Borel measurable subsets of Ω. The observation is denoted $r(t,\omega)$, $t\in E$, $\omega\in\Omega$ where under hypothesis j, $r(t,\omega)=x_j(t,\omega)$. We use the Lebesque decomposition to write Ω as a disjoint union of measurable subsets $\{\Omega_\rho,\Omega_\nu,\Omega_{\rho\nu}\}$, such that $\rho(\Omega_\nu)=\nu(\Omega_\rho)\equiv 0$ and such that $\rho\approx\nu$ when restricted to $\Omega_{\rho\nu}$. The logarithm of the likelihood functional, $\Lambda = \Lambda[r:t \in E, \omega \in \Omega]$ may then be defined by

[illegible] $r \in \Omega_\rho$

[illegible] $r \in \Omega_{\rho\nu}$

[illegible] $r \in \Omega_\nu$

The test for discriminating between ρ and ν is to choose ρ if Λ is greater than or equal to a threshold L, and to choose ν otherwise.

A direct measure of the performance of this test is to calculate the probabilities of an error of the first and second kind, P_{12} and P_{21}, defined as

$$P_{12} = Pr[\text{choose } H_1 | H_2 \text{ true}] = \int_L^\infty d\mu[\Lambda | H_2]$$

$$P_{21} = Pr[\text{choose } H_2 | H_1 \text{ true}] = \int_{-\infty}^L d\mu[\Lambda | H_1]$$

where μ is the probability distribution of the log likelihood functional. An indirect measure of performance is the total probability of error, P_E, defined as

$$P_E = \pi_1 P_{21} + \pi_2 P_{12}$$

where π_j is the *a priori* probability that hypothesis j is true.

H. Chernoff [4] was apparently one of the first in the mathematical statistics literature to obtain the following simple upper bounds on the probabilities of an error of the first and second kind:

$$P_{21} \leq \inf_{s<1} h_s(\rho,\nu) e^{-(s-1)L}$$

$$P_{12} \leq \inf_{s>0} h_s(\rho,\nu) e^{-sL}$$

where $h_s(\rho,\nu)$ is a conditional moment generating function associated with the distribution of the log likelihood functional,

$$h_s(\rho,\nu) = E[e^{s\Lambda} | H_2] = E[e^{(s-1)\Lambda} | H_1]$$

An alternate expression for $h_s(\rho,\nu)$ is

$$h_s(\rho,\nu) = \int (d\rho/d\kappa)^s (d\nu/d\kappa)^{(1-s)} d\kappa \quad \rho,\nu \ll \kappa$$

which is independent of the choice of κ.

These ideas have been extended in several different directions. Shannon

et al. [15] obtained lower bounds on the probabilities of an error of the first and second kind, for discrete measures ρ,ν and showed their lower bounds and the Chernoff upper bounds are identical to within a factor that approaches unity as the observation becomes infinite.

It is well known [9] that evaluating $h_{1/2}(\rho,\nu)$ yields simple, easy to calculate (but often crude) upper and lower bounds on the total probability of error,

$$\tfrac{1}{2}\min(\pi_1,\pi_2)[h_{1/2}(\rho,\nu)]^2 \leq P_E \leq (\pi_1\pi_2)^{1/2} h_{1/2}(\rho,\nu)$$

where the threshold L equals $\ln(\pi_2/\pi_1)$ in order to obtain the upper bound on P_E.

We emphasize that in many cases of practical interest, it is difficult or impossible to obtain closed form analytic expressions for P_{12}, P_{21}, P_E, and, furthermore, it is also quite difficult or expensive to accurately numerically approximate these quantities. The bounds just discussed are often possible to treat both analytically and numerically. Thus, the Kakutani product $h_s(\rho,\nu)$ provides upper and lower bounds on P_E immediately, and can be further manipulated to obtain Chernoff-type bounds on P_{12}, and P_{21}. Although we are finding bounds on a likelihood test, many of the same ideas developed here can be extended to bound performance of sub-optimum tests, or, equivalently, to assess the performance of a sub-optimum versus optimum test.