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EXPLICIT SOLUTIONS TO SOME SINGLE-PERIOD INVESTMENT PROBLEMS FOR RISKY LOG-STABLE STOCKS

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Numerical approximations are presented for the expected utility of wealth over a single time period for a small investor who proportions her or his available capital between a risk-free asset and a risky stock. The stock price is assumed to be a log-stable random variable. The utility functional is logarithmic or isoelastic (y^q/q, $q < 0$). Analytic results are presented for special choices of model parameters, and for large and small time periods.

1. Introduction

Over the past decade statistical evidence has been accumulated which indicates prices of certain stocks can be adequately modeled as a geometric random walk: logarithms of successive prices, when differenced, appear to be adequately modeled as independent identically distributed random variables [e.g., Mandelbrot (1963), Fama (1965a, 1965b, 1970), Fielitz (1972, 1976), Blattberg and Gonedes (1974), Barnea and Downes (1973), Paulson et al. (1975), Leitch and Paulson (1975), and the review by Samuelson (1973)]. The underlying probability distribution is known to be distinctly non-Gaussian [e.g., Fama (1965a), Leitch and Paulson (1975)]; some of the evidence available at present indicates this distribution can be adequately modeled using stable distributions, distributions which arise from the central limit theorem of probability theory [e.g., Fama (1965a), Fielitz (1972, 1976), Paulson et al. (1975), Leitch and Paulson (1975)]. For well established companies the characteristic index α of the underlying stable distribution lies in the neighborhood of 1.8 to 1.9 [e.g., Fama (1965a) Leitch and Paulson (1975)], close to the Gaussian characteristic index of 2.0.

Unfortunately, the non-Gaussian stable distributions have not been as extensively studied as the Gaussian distributions in the mathematical literature [suitable references on stable distributions are Feller (1966), Gnedenko and Kolmogorov (1968), and Lukacs (1970)]. For example, at present only power series expansions are known for stable probability density functions, and in only three cases (one of which is the Gaussian) do these series reduce to closed form analytic expressions. Furthermore, in order to treat the many interesting problems associated with stock portfolios [e.g., Markowitz (1959), Fama (1965b), Samuelson (1967a, 1967b), Ohlson (1972)], it is necessary to have an adequate theory of multivariate stable distributions, and again this machinery only exists in the Gaussian case. These two problems, as well as others, have probably hindered investors in applying the log-stable random walk model in investment strategies, and have led to renewed interest in models with time varying parameters [e.g., Barnea and Downes (1973)] and in models with analytically tractable probability distributions [e.g., Blattberg and Gonedes (1974)]. This issue will be briefly revisited later in this work.

Here a well known problem [Mossin (1968), Samuelson (1969), Hakansson (1970)] in finance theory is examined, where an investor must decide what proportion of available capital should be invested in a risky log-stable stock versus in a risk-free asset with a guaranteed rate of return. Within this narrow framework a number of interesting new results are obtained.

Throughout, the emphasis is on using numerical methods to study the qualitative nature of the solution to a simple problem [see also Ziemba et al. (1974) for work similar in spirit]. The issues addressed here are where the maximum of the expected utility functional occurs as a fraction of capital invested in the risky asset, and how flat the expected utility is in the neighborhood of its maximum. The absolute of the expected utility functional at its maximum is not of immediate interest, since the conclusions reached should be independent of any affine transformation of the utility. For this reason, the reader should keep in mind that no scales are shown for the expected utility functional in the figures to follow.

One reason for using numerical methods is that analytic methods appear to lead to insurmountable algebriac complexity and obscure basic features of the expected utility functional as a function of the various parameters in the problem (at least for the problems studied here). A second reason for attempting this is that numerical results often tend to give substance to theory, and sometimes suggest unsuspected avenues for constructive research. These reasons were the primary motivation for this work.

The paper is outlined as follows. In the next section the problem is precisely formulated. The following section discusses the results of numerically approximating the solution to the problem, and develops analytic results in special cases. An appendix details the numerical methods used in approximating the expected utility functionals.

2. Problem statement

The statement of the problem is heavily influenced by earlier work of Mossin (1968), Samuelson (1969), and Hakansson (1970).

Let x_k $(k = 1, 2, \ldots)$ be the price of the stock at time $k\Delta t$, where Δt is a given interval of time. The price is assumed to evolve in time according to the formula

$$x_{k+1} = e^{s_k} x_k, \qquad x_0 \text{ given}, \quad k = 0, 1, 2, \ldots, \tag{1}$$

where $\{s_k\}$ is a sequence of independent identically distributed random variables. An immediate consequence of this model is

$$x_k = e^s x_0, \qquad k = 1, 2, \ldots, \tag{2}$$

$$s \equiv \sum_{i=0}^{k-1} s_i. \tag{3}$$

The investor is assumed to be a small investor: the amount of stock the investor purchases has no effect on the stock price. It is also assumed (for simplicity) that the parameters that specify the probability distribution of s_k are known, although in a more realistic problem the parameters are unknown and must be estimated.

Initially the investor allocates a fraction p of available capital W_0 to the risky stock, and a fraction $(1-p)$ to the risk-free asset. At time $k\Delta t$, the total wealth of the investor is W_k,

$$W_k = (1+r)(1-p)W_0 + pe^s W_0, \tag{4}$$

$$W_k = [(1-p)+p\, e^{s-R}]W_0 \cdot e^R, \tag{5}$$

where $(1+r) \equiv e^R$ is the return on the risk-free asset at time $k\Delta t$. Both r and R will be called the rate of return from this point on. The investor quantifies preferences according to a utility function U, which is assumed measurable and sufficiently smooth, with positive first derivative and negative second derivative [see Arrow (1971)]. Since the investor can specify the investment period, without loss of generality k can be taken to unity in (4), (5).

If the investor wishes to maximize expected wealth at the end of the investment period, and cash in all investments afterwards, then the problem is to find that value of p, say p_{opt}, to invest in the risky stock such that the expected utility of wealth W_1 is maximized while $W_2 = 0$,

$$p_{opt}: \max_{0 \leqq p \leqq 1} E[U(W_1)], \qquad W_2 = 0. \tag{6}$$

By a simple argument [see Samuelson (1969)], this is entirely equivalent to maximizing expected utility of consumption, where consumption takes place at the beginning of the time interval.

The generalization of this to a multi-period problem involves allowing the investor to buy and sell stock at the end of each period, investing a different fraction of wealth into the stock after each time interval while attempting to maximize the expected utility functional, with a specified terminal state. Since dynamic programming reduces this multi-period problem to a squence of single-period problems, and since few explicit solutions are presently available for the single-period problem, it was felt a good grasp of the single-period problem was prerequisite to tackling the more difficult multi-period problem.

The extension of this work from one risky asset and one risk-free asset to a portfolio of risky assets and one risk-free asset is immediate from a separation theorem [Cass and Stiglitz (1970)] whose hypotheses are valid here. An important qualification is the portfolio of risky assets must have a log-stable distribution, which obviously does not imply that the marginal distributions of each of the risky assets are log-stable [cf. Markowitz (1959)].

Mossin (1968), Samuelson (1969) and Hakansson (1970) among others [e.g., Merton (1969, 1971, 1973)] have observed that the particular choice of initial wealth and consumption do not affect the solution to the single-period (or multi-period) problem if and only if the investor's preferences can be modeled by isoelastic utility functions. For this reason, as well as for their simple analytic form, this class of utility functions has received a great deal of attention. Two isoelastic utility functions that will be examined in detail are the Bernoulli logarithmic utility,

$$\begin{aligned} U(y) &= \ln(y), & y &> 0, \\ &= 0, & y &\leqq 0, \end{aligned} \tag{7}$$

and the power utility,

$$\begin{aligned} U(y) &= y^q/q, & y &> 0, \\ &= 0, & y &\leqq 0, \end{aligned} \tag{8}$$

where $q < 0$ (this restriction on q being negative is necessary to insure the existence of the expected utility of wealth for the non-Gaussian log-stable model). As is well known, the logarithmic utility is an exceptional member of this family, since $U(y) = q^{-1}(y^q - 1) \to \ln(y)$ as $q \to 0$.

The assumption of no brokerage commissions can be relaxed in many ways. One way of achieving this is as follows. Assume a net gain or loss G on the stock investment at the end of the investment period. The amount lost on brokerage commissions, i.e., the amount subtracted from wealth in order to actually achieve this gain or loss is $\hat{c}|G|$, where $\hat{c}$ is the commission. On the other hand, the fraction invested in the stock should be modified up or down to take this into account: the correct fraction to invest in the stock is the sum of the final price of the stock plus the gain or loss divided by the final wealth, or

$$p = \frac{p\,\mathrm{e}^{s-R}W' + G}{[(1-p) + p^s\mathrm{e}^{s-R}]W'}, \tag{9}$$

where $W' \equiv \mathrm{e}^R W_0$. Solving for G and substituting into the expected utility, the problem is to find p such that

$$E\{U[\{(1-p)+p\,e^{s-R}-\hat{c}p(1-p)|e^{s-R}-1|\}W']\} \tag{10}$$

is maximized over p.

It remains to specify the probability distribution of the logarithm of successive differences of stock prices. One rich class of probability distributions that have been studied extensively in the mathematical literature are the infinitely divisible distributions; the characteristic function of an infinitely divisible distribution can always be written in a simple canonical form due to Lévy and Khinchin [Gnedenko and Kolmogorov (1968), Lukacs (1970)]

$$E(e^{iws}) = \exp\left(i\delta w-\tfrac{1}{2}\sigma^2 w^2+\int_{v\neq 0}\left(e^{ivw}-1-\frac{ivw}{1+v^2}\right)d\mu(v)\right), \tag{11}$$

where $\delta(-\infty < \delta < \infty)$ is a location parameter, $\sigma^2(\geqq 0)$ is the variance of the Gaussian component, and μ is called the Lévy measure which specifies the parameters of a generalized Poisson distribution, with

$$\int_{v\neq 0}\frac{v^2}{1+v^2}\,d\mu(v) < \infty,$$

and $d\mu$ non-negative.

To date, attention has centered on the purely Gaussian distribution ($\mu = 0$) and on one particular non-Gaussian distribution ($\sigma^2 = 0$), where $\mu(v) = c_-|v|^{-\alpha}$ for $v < 0$, and $\mu(v) = -c_+v^{-\alpha}$ for $v > 0$, which is a stable distribution with characteristic index α, $0 < \alpha < 2$, and skewness parameter β, $\beta = (c_- - c_+)/(c_+ + c_-)(1 \leqq \alpha < 2)$. The reasons for examining these two are that first, they are the only distributions which arise from the central limit theorem of probability theory, and thus offer some hope of grossly accounting for all the factors that influence stock price fluctuations at one sweep [e.g., see Samuelson (1973)], and second, these distributions have been successfully fit to actual stock price data [Fama (1965a), Paulson et al. (1975), Leitch and Paulson (1975)].

Clearly, the class of infinitely divisible distributions include many other cases than just these two. If one wishes to model stock price fluctuations using a gamma distribution, for example, then $\sigma^2 = 0$, $d\mu = ae^{-bv}\,d(\ln v)$. If one wishes to use models with time varying parameters, then simply let (δ, σ^2, μ) depend on time. The Gaussian and non-Gaussian stable distribution models employed in the next section provide a convenient starting point for investigating these other models, in that they capture much of the behavior of Gaussian and non-Gaussian infinitely divisible distributions.[1]

[1] Much of the literature is concerned with the tail behavior of $P(s)$. Simple arguments [Feller (1966, p. 540)] show that if $\int_x^\infty d\mu(v) \sim 0(x^{-r})$, then $\Pr(s > x) \sim 0(x^{-r})$, while if μ has compact support, then s has moments of all orders. Thus, infinitely divisible distributions can model a broad range of tail behavior of $P(s)$, by simply choosing the Lévy measure appropriately.

3. Numerical approximations and analytic results

In this section the results of numerically approximating the expected utility function,

$$E(U) = \int_{-\infty}^{\infty} U[\{(1-p)+p\,e^{s-R}\}W']f(s)\,ds, \tag{12}$$

are described for the Bernoulli logarithmic utility $[U(y) = \ln y]$ and for the isoelastic utility $[U(y) = y^q/q, q < 0]$, where $f(s)$ is the probability density function of a stable distribution with characteristic index α, skewness parameter β, dispersion parameter γ, and location parameter δ. Brokerage commissions have been set to zero here; machine calculations to be discussed later indicate brokerage commissions are not significant in this problem, although they may well be in a more complicated, realistic problem.

Before presenting the results, it is worthwhile to digress and develop some intuition for the behavior of a representative sample function of a utility functional. Figs. 1 and 2 depict representative sample functions of arithmetic symmetric stable random walks [see eq. (3) for the definition of an arithmetic random walk], as well as the corresponding logarithmic utility sample functions with $R = 0$, $p = \frac{1}{2}$. The large hops in the non-Gaussian stable random walks are a graphic indication that a sum of infinite variance random variables is often dominated by one or a few of the summands; these large excursions allow stable distribution models to account for such sudden jumps in stock prices such as occurred in 1962 or 1973, as well as many other instances. These large hops would be absent in a Gaussian random walk. Note the logarithmic utility is bounded from below by $\ln(1-p)$, while there is no upper bound. It is clear from these pictures that the stable random walk with $\alpha = 1.9$ does not wander as far positive as the random walk with $\alpha = 1.5$, which reflects the fact the stable distribution with $\alpha = 1.9$ has less probability in its tails than the $\alpha = 1.5$ stable distribution. Observe that around $t \approx 450$ the utility functional drops to zero in all cases. One cure for this is to assume there is a positive long-term drift upward in the random walk (the pictures were generated assuming zero drift), which would modify the revivals at $t \approx 650$ and $t \approx 750$.

Various analytical results are now presented for special choices of model parameters, and for long- and short-time periods. For ease of exposition, only two cases are studied in detail, since the other results follow in like manner (all analytic results are collected in table 1).

Proposition. Assume the following conditions are satisfied:

(a) $U(y) = \ln(y)$,

(b) $\delta = R$, *i.e., the drift in the stock price equals the rate of return on the risk-free asset.*

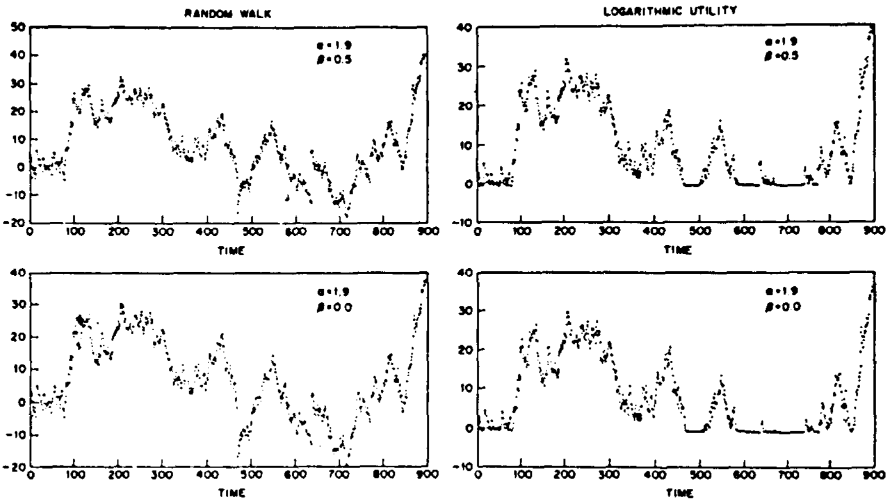

Fig. 1. (a) Left figures (random walk versus time): $S_{n+1} = S_n + X_n$, $S_0 = 0$, $\{X_n\}$ independent identically distributed stable random variables. Stable distribution parameters ($\alpha = 1.9$, β, $\gamma = 1$, $\delta = 0$). (b) Right figures (logarithmic utility versus time): $U_n = \ln[pe^{S_n} + (1-p)e^R]$, p = fraction invested in stock = $\frac{1}{2}$, R = rate of return on risk-free asset = 0.

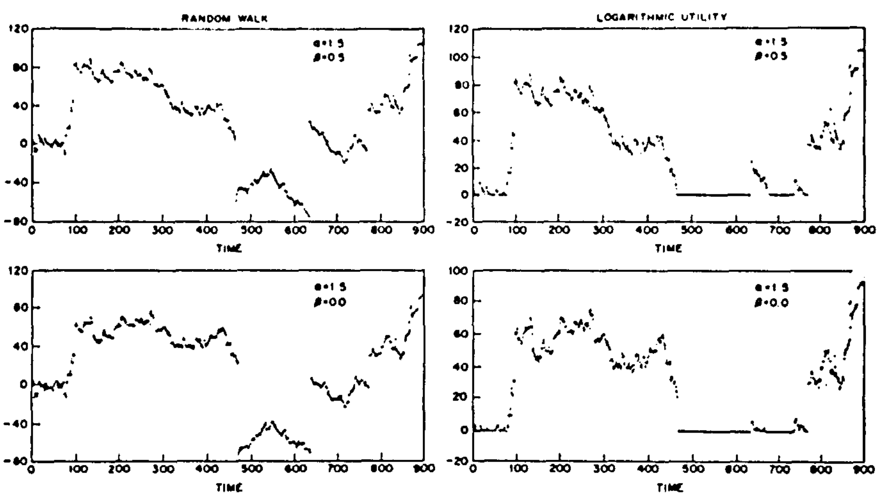

Fig. 2. (a) Left figures (random walk versus time): $S_{n+1} = S_n + X_n$, $S_0 = 0$, $\{X_n\}$ independent identically distributed stable random variables. Stable distribution parameters ($\alpha = 1.5$, β, $\gamma = 1$, $\delta = 0$). (b) Right figures (logarithmic utility versus time): $U_n = \ln[pe^{S_n} + (1-p)e^R]$, p = fraction invested in stock = $\frac{1}{2}$, R = rate of return on risk-free asset = 0.

Then the following holds:

(i) *If* $f(s) = f(-s)$, *i.e., if the distribution is symmetric (not necessarily stable),* $p_{opt} = \frac{1}{2}$.

(ii) *If* $f(s)$ *is a stable density with characteristic exponent* α *and skewness parameter* β, *and the duration of the investment interval is infinite, then*

$$p_{opt} = \tfrac{1}{2}\left(1 - \beta' \cdot \frac{2-\alpha}{\alpha}\right), \qquad 1 < \alpha \leqq 2,$$

where

$$\tan\left(\beta' \cdot \frac{\pi}{2}\frac{2-\alpha}{\alpha}\right) = -\beta \tan(\pi\alpha/2).$$

Table 1
Optimum fraction to invest in a risky asset.

I. *Investment period approaches zero duration:*

$$0 \leqq p_{opt} \leqq 1.$$

II. *Investment period approaches infinite duration:*

(a) Bernoulli logarithmic utility, $U(y) = \ln(y)$:

(i) $\delta < R$, $p_{opt} = 0$,

(ii) $\delta = R$, $p_{opt} = \frac{1}{2}\left(1 - \beta' \frac{2-\alpha}{\alpha}\right)$, $\tan\left(\beta' \frac{\pi}{2}\frac{2-\alpha}{\alpha}\right) = -\beta \tan\left(\frac{\pi\alpha}{2}\right)$,

(iii) $\delta > R$, $p_{opt} = 1$.

(b) Isoelastic utility, $U(y) = y^q/q$:

$$p_{opt} = 0.$$

III. *Symmetric distribution,* $f(s) = f(-s)$, $U(y) = \ln(y)$:

$$p_{opt} = \tfrac{1}{2}.$$

The proof of this proposition is omitted since it is straightforward. Statement (i) is easy to show from simply computing the derivative of the expected utility functional with respect to p; note that this includes the case of a student t model for stock price fluctuations [Blattberg and Gonedes (1974)]. Statement (ii) follows the same line of reasoning, and uses results of Zolotarev (1966). Table 2 summarizes representative values of $p_{opt} = p_{opt}(\alpha, \beta)$; only one half the table is shown, because $p_{opt}(\alpha, \beta) + p_{opt}(\alpha, -\beta) \equiv 1$. As a simple check, note that for $\beta = -1$, $p_{opt} = (1 - 1/\alpha)$, while for $\beta = 1$, $p_{opt} = 1/\alpha$.

The discussion now turns to numerical approximation of the expected utility functional. The approximations used are discussed in an appendix. In all cases,

Table 2

Optimum fraction to invest in a risky stock to maximize expected logarithmic utility functional for $\delta_0 = R_0$, with an infinite investment period. $\delta_0 = \lim_{t\to\infty}(\delta/t)$, $R_0 = \lim_{t\to\infty}(R/t)$, δ_0 is the log-stable stock price drift (see appendix), R_0 is the rate of return on the risk-free asset.

α \ β	-1.00	-0.75	-0.50	-0.25
1.95	0.48718	0.49038	0.49358	0.49679
1.90	0.47368	0.48019	0.48676	0.49337
1.80	0.44444	0.45773	0.47152	0.48567
1.70	0.41176	0.43165	0.45329	0.47628
1.60	0.37500	0.40074	0.43068	0.46425
1.50	0.33333	0.36344	0.40161	0.44801
1.40	0.28571	0.31782	0.36295	0.42465
1.30	0.23077	0.26150	0.31000	0.38831
1.20	0.16667	0.19178	0.23619	0.32604
1.10	0.09091	0.10568	0.13423	0.20887

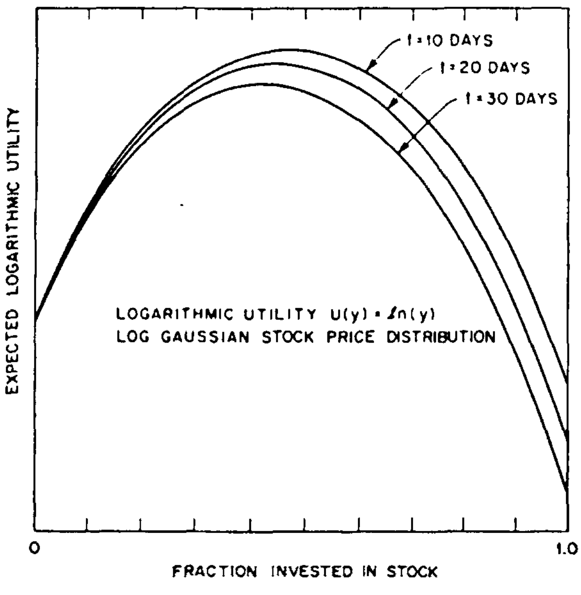

Fig. 3. Expected logarithmic utility $E(U)$ versus fraction invested in stock. $E(U) = \int_{-\infty}^{\infty} f(S,t) \ln[(1-p)+pe^{S-R_0 t}]\,dS$, $f(S,t) = f(\alpha = 2, \beta, \gamma = \gamma_0 t, \delta = \delta_0 t; S) = \exp[-(S-\delta_0 t)^2/4\gamma_0 t]/(4\pi\gamma_0 t)^{\frac{1}{2}}$, $R_0 = \delta_0$, $t = 10, 20, 30$ days. Representative values for γ_0 taken from Fama (1965a, table 5). R_0 is rate of return on risk-free asset, δ_0 is drift in log Gaussian stock price model.

representative values were taken for the value of the dispersion of the distribution of various conservative blue-chip stocks traded on the New York Stock Exchange [Fama (1965a, table 5)].

Fig. 3 shows representative expected log utility functionals for investment periods of ten, twenty and thirty days as a function of p, where $f(s)$ is a Gaussian density ($\alpha = 2$). In all cases the expected utility is maximized for $p = 0.5$. Note that the expected log utility is relatively flat in the neighborhood of its optimum, so that investing a much smaller (and larger) amount in the stock yields roughly the same expected utility.

Next, the results for $f(s)$ a non-Gaussian stable density ($1 < \alpha < 2$) are described. Fig. 4 shows representative expected logarithmic utility functionals as functions of both the fraction invested in the stock and the duration of the

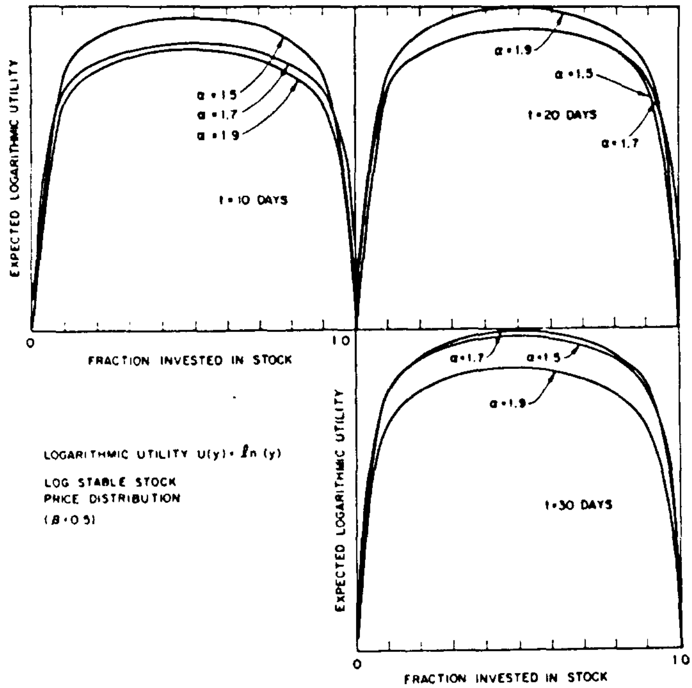

Fig. 4. Expected logarithmic utility $E(U)$ versus fraction invested in stock. $E(U) = \int_{-\infty}^{\infty} f(S,t)\ln[(1-p)+pe^{S-R_0 t}]\,dS$, $f(S,t) = f(\alpha,\beta,\gamma=\gamma_0 t,\delta=\delta_0 t; S)$, $R_0 = \delta_0$, $t = 10$, 20, 30 days. Representative values for γ_0 taken from Fama (1965a, table 5). R_0 is rate of return on risk-free asset, δ_0 is drift rate in log-stable stock price model.

investment interval. Note that for fixed time the expected utility is quite flat as a function of p. The reader is cautioned that different vertical scales have been used for ezch curve, in order to convey as much qualitative information in as short a space as possible. In each case the expected log utility is maximized for $p \approx 0.5$. As in the log Gaussian case, the expected utility is quite flat in the neighborhood of its optimum, so a much smaller (or larger) investment yields only marginally less than the optimum expected utility. The effect of varying the characteristic index upon the expected log utility will be dealt with elsewhere.

In addition to the results shown in figs. 3 and 4 expected logarithmic utilities were approximately calculated with the dispersion varying up and down by a factor of two, varying the rate of return on the risk-free asset from 0 percent to 10 percent annually, varying the drift of the stock price from -10 percent to 10 percent annually, and choosing brokerage commissions of 0 percent, 1 percent, and 5 percent. All these variations had negligible effect on the main

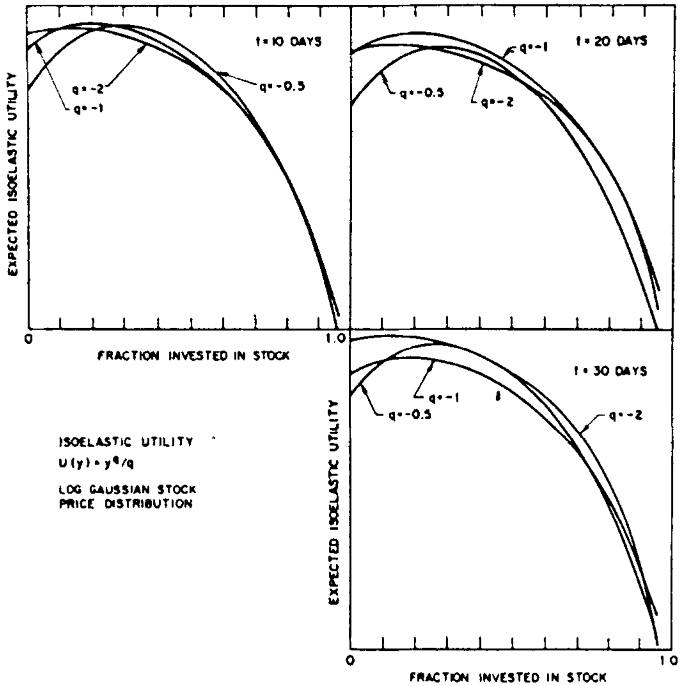

Fig. 5. Expected isoelastic utility $E(U)$ versus fraction invested in stock. $E(U) = \int_{-\infty}^{\infty} \mathrm{d}S \cdot f(S,t)[(1-p)+pe^{S-R_0 t}]^q/q$, $f(S,t) = f(\alpha = 2, \beta, \gamma = \gamma_0 t, \delta = \delta_0 t; S) = \exp[-(S-\delta_0 t)^2/4\gamma_0 t]/(4\pi\gamma_0 t)^{1/2}$, $R_0 = \delta_0$; $t = 10, 20, 30$ days; $q = -0.5, -1.0, -2.0$. Representative values for γ_0 taken from Fama (1965a, table 5). R_0 is rate of return on risk-free asset, δ_0 is drift rate parameter in log Gaussian stock price model.

results: in all cases the optimum fraction was around $p \approx 0.5$, and the expected utility in the neighborhood of the optimum was relatively flat.

When $f(s)$ is Gaussian and the utility is isoelastic, $U(y) = y^q/q$, the results are quite similar. Fig. 5 shows various numerical approximations to (12) as a function of p ($q = -0.5, -1.0, -2.0$; investment periods of 10, 20 and 30 days), q is fixed in each graph, investment period is a parameter. Again, the reader is cautioned different scales have been used for each curve in order to emphasize the optimum. The same parameters as used in the logarithmic utility have been used here. Note that the optimum p is around 0.4 for $q = -0.5$ and decreases to

0.1 for $q = -2$. Close inspection reveals the optimum decreases as time increases, as expected. Fig. 6 shows the same information as in fig. 5, except that time (t) is fixed in each graph. These figures make it evident that investing much more or less in the risky stock yields only slightly less expected utility than the optimum fraction.

Varying the rate of return on the risk-free asset from 0 percent to 10 percent annually, changing the drift of the stock price from -10 percent to 10 percent

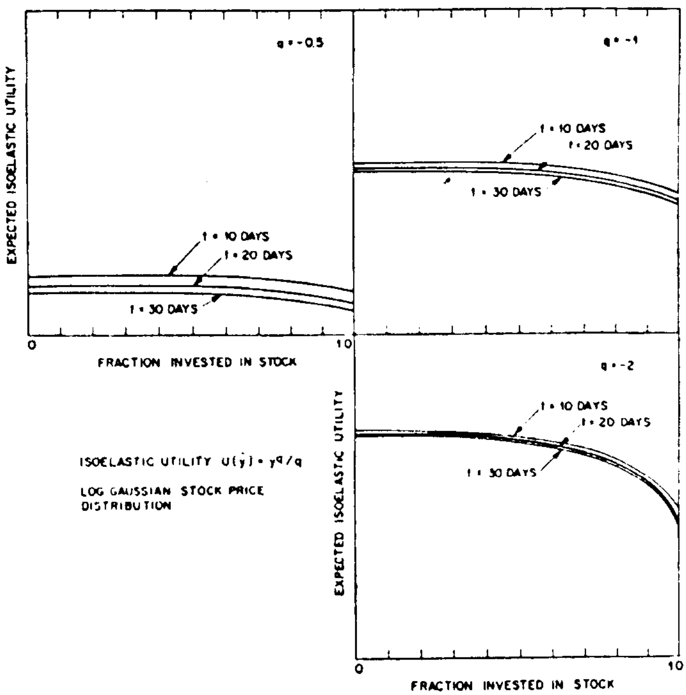

Fig. 6. Expected isoelastic utility $E(U)$ versus fraction invested in stock. $E(U) = \int_{-\infty}^{\infty} \mathrm{d}S \cdot f(S,t)[(1p)+pe^{S-R_0 t}]^q/q$, $f(S,t) = f(\alpha = 2, \quad \beta, \quad \gamma = \gamma_0 t, \quad \delta = \delta_0 t; S) = \exp[-(S-\delta_0 t)^2/4\gamma_0 t]/(4\pi\gamma_0 t)^{\frac{1}{2}}$, $R_0 = \delta_0$; $t = 10, 20, 30$ days; $q = -0.5, -1.0, -2.0$. Representative values for γ_0 taken from Fama (1965a, table 5). R_0 is rate of return on risk-free asset, δ_0 is drift rate parameter in log Gaussian stock price model.

annually, increasing the dispersion by a factor of two and also decreasing it by a factor of two, and switching the brokerage commission from 0 percent to 1 percent to 5 percent had virtually no effect on these results.

The final case is when $f(s)$ is non-Gaussian stable, and $U(y) = y^q/q$.

Fig. 7 shows representative expected isoelastic utility functionals as a function of both the fraction invested in the stock and the duration of the investment interval. The flatness of the expected utility in the neighborhood of its maximum is quite evident in these figures. The only curvature evident is in the neighborhood of p equals one. The optimum fraction is apparently around 0.4 for $q = -0.5$, and decreases to perhaps 0.1 for $q = -2$. Furthermore, the expected utility is quite close to its maximum for fractions much less (but not greater)

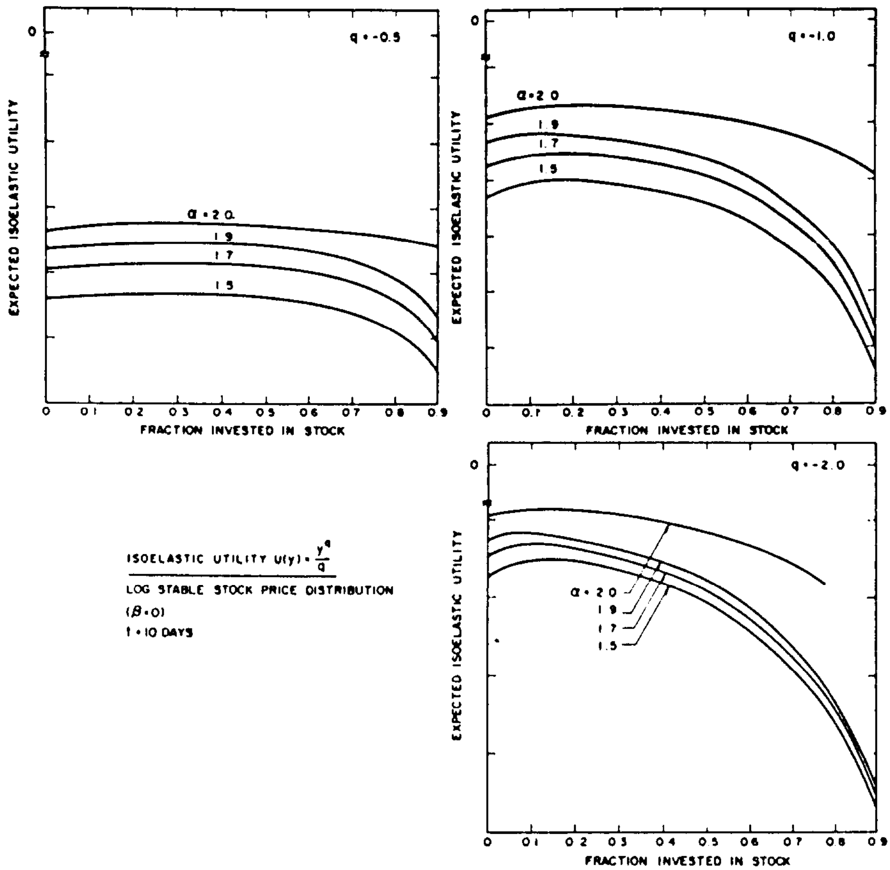

Fig. 7. Expected isoelastic utility $E(U)$ versus fraction invested in stock. $E(U) = \int_{-\infty}^{\infty} dS\cdot f(S,t)[(1-p)+pe^{S-R_0 t}]^q/q, f(S,t) = f(\alpha, \beta, \gamma = \gamma_0 t, \delta = \delta_0 t; S), R_0 = \delta_0 = 0, t = 10$ days; $q = -0.5, -1.0, -2.0$. Representative values for γ_0 taken from Fama (1965a, table 5). R_0 is rate of return on risk-free asset, δ_0 is drift rate parameter in log-stable stock price model.

than the optimum, as in the log Gaussian model. The effect of varying the characteristic index upon the expected isoelastic utility is of independent interest, and will be dealt with elsewhere.

Varying dispersion, drift of the stock price, rate of return on the risk-free asset, and including brokerage commissions in the manner described earlier had virtually no effect on these results.

Appendix

Numerical analysis

In this section the approximations used to numerically evaluate the integral

$$J = \int_{-\infty}^{\infty} U[(1+r)(1-p)+p\,e^{s}]f(s;\alpha,\beta,\gamma,\delta)\,ds \tag{A.1}$$

are described, for the Bernoulli logarithmic utility $[U(y) = \ln y]$ and for the isoelastic utility $[U(y) = y^q/q, q < 0]$.

The first step is to change variables

$$s' = (s-\delta)/c, \tag{A.2}$$

$$\therefore J = \int_{-\infty}^{\infty} U[(1+r)(1-p)+p\,e^{cs'+\delta}]f(s';\alpha,\beta,\gamma = 1,\delta = 0)\,ds', \tag{A.3}$$

and the prime superscript on s' will be dropped from here on. The integral is now broken up into three integrals,

$$J = J_1+J_2+J_3, \tag{A.4}$$

with respective limits $(-\infty, -L)$, $(-L, M)$, and (M, ∞). The integrand in J_1 can be simplified by series expansions of the utility function. For the Bernoulli logarithmic utility,

$$\ln[(1+r)(1-p)+p\,e^{cs+\delta}] = \ln[(1+r)(1-p)] - \sum_{j=1}^{\infty}\left(\frac{p\,e^{\delta}}{(1+r)(p-1)}\right)^{j} e^{jsc}.\frac{(-1)^{j+1}}{j}, \tag{A.5}$$

while for the isoelastic case,

$$[(1+r)(1-p)+p\,e^{sc+\delta}]^q/q = \frac{[(1+r)(1-p)]^q}{q}\left\{1+\frac{qp}{(1+r)(1-p)}e^{sc+\delta}+\frac{q(q-1)}{1\cdot 2}\times\left[\frac{e^{sc+\delta}p}{(1+r)(1-p)}\right]^2+\ldots\right\}. \tag{A.6}$$

For both cases, for L sufficiently large, only a small number of terms are needed to accurately approximate the integrand. Two approaches were used to evaluate J_1, one a straightforward Romberg[2] adaptive step size integration method

[2]Ralston (1965) is a highly readable reference on Romberg numerical integration schemes.

using an asymptotic series expansion for the stable density in conjunction with (A.5)–(A.6), the second a laborious integration by parts of the asymptotic series expansion for the density along with (A.5)–(A.6). Both methods yielded answers consistent to three significant figures and were less than the obvious upper bound on J_1,

$$J_1 \leqq U[(1+r)(1-p)+p\,e^{-cL+\delta}]\cdot \Pr(s < -L). \tag{A.7}$$

The second integral, J_2, was evaluated two different ways. The first approach used a Romberg adaptive step size integration method; the stable density was evaluated via power and asymptotic series. The second approach used a Romberg fixed step size integration method, where the stable density was approximated by inverting the characteristic function using the discrete fast Fourier transform. Both approaches yielded results consistent to three significant figures.

J_3, the third integral, can be found by expanding the utility in a power series. For the Bernoulli logarithmic utility,

$$\ln[(1+r)(1-p)+p\,e^{sc+\delta}] = (\ln p)+sc+\delta + \ln\left[1+\frac{(1+r)(1-p)}{p}e^{-sc-\delta}\right], \tag{A.8}$$

$$\ln\left[1+\frac{(1+r)(1-p)}{p}e^{-sc-\delta}\right] = -\sum_{j=1}^{\infty}\left[\frac{(1+r)(p-1)}{p}e^{-\delta}\right]^j e^{-jsc}/j, \tag{A.9}$$

$$J_3 = (\ln p+\delta)\Pr[s > R]+\int_{\infty}^{M} cs f(s;\alpha,\beta,\gamma=1,\delta=0)\,ds - \int_M^{\infty}\sum_{j=1}^{\infty}\left[\frac{(1+r)(p-1)}{p}\right]^j e^{-jsc}/j\, f(s;\alpha,\beta,\gamma=1,\delta=0)\,ds. \tag{A.10}$$

The first term in (A.10) can be computed directly from series expansions for the distribution. The second term in (A.10) was found by direct integration of the density asymptotic series multiplied by s. The last term in (A.11) was approximated by both a Romberg adaptive step size algorithm and by a laborious integration by parts of the log power series times the stable density asymptotic series; both methods yielded answers consistent to three significant figures, and were checked against the upper bound,

$$\int_M^{\infty}\ln\left[1+\frac{(1+r)(1-p)}{p}e^{-sc-\delta}\right]f(s;\alpha,\beta,\gamma=1,\delta=0)\,ds \leqq \ln\left[1+\frac{(1+r)(1-p)}{p}e^{-cM-\delta}\right]\Pr[s > M]. \tag{A.11}$$

For the isoelastic utility,

$$[(1+r)(1-p)+p\,\mathrm{e}^{sc+\delta}]^q/q = \frac{(p\,\mathrm{e}^{sc+\delta})^q}{q}\left(1+\frac{(1+r)(1-p)}{p}\,\mathrm{e}^{-sc-\delta}\right)^q, \tag{A.12}$$

$$\left[1+\frac{(1+r)(1-p)}{p}\,\mathrm{e}^{-sc-\delta}\right]^q = 1+\frac{q(1+r)(1-p)}{p}\,\mathrm{e}^{-sc-\delta} + \frac{q(q-1)}{2}\left[\frac{(1+r)(1-p)}{p}\,\mathrm{e}^{-sc-\delta}\right]^2+\dots, \tag{A.13}$$

$$J_3 = \int_M^\infty \frac{1}{q}(p\,\mathrm{e}^{sc+\delta})^q\left[1+\frac{q(1+r)(1-p)}{p}\,\mathrm{e}^{-sc-\delta}+\frac{q(q-1)}{2}\left[\frac{(1+r)(1-p)}{p}\,\mathrm{e}^{-sc-\delta}\right]^2+\dots\right] \times f(s;\alpha,\beta,\gamma=1,\delta=0)\,\mathrm{d}s. \tag{A.14}$$

The integral (A.14) was evaluated using both a Romberg adaptive step size algorithm and a tedious integration by parts of the utility function series expansion (A.12)-(A.13) times the stable density asymptotic series; both methods yielded answers consistent to three significant figures and were checked against the upper bound,

$$\left|\int_M^\infty \frac{1}{q}[(1+r)(1-p)+p\,\mathrm{e}^{sc+\delta}]^q f(s;\alpha,\beta,\gamma=1,\delta=0)\,\mathrm{d}s\right| \leqq \left|\frac{1}{q}\right|[(1+r)(1-p)+p\,\mathrm{e}^{cM+\delta}]^q\,\Pr[s>M]. \tag{A.15}$$

In all cases, an absolute error criteria of 0.001 was used for the Romberg integration methods.

To give the reader some feeling for the computational effort involved in generating the plots in the text, some numbers for a typical run are now described. An average run involved one hundred evaluations of the expected utility function (p varied from 0.001 to 0.999 in nine evenly spaced steps, time varied from 0.01 to 30.0 days in nine evenly spaced intervals). On a Honeywell 6000 computer 24K of core storage would be needed, and this run would take 78 seconds. Over one hundred and fifty runs were carried out to gain confidence in the approximations involved in numerically evaluating the expected utility functional. The figures presented in the text are felt to be representative, and all errors involved appear to be negligible.

S.O. Rice has suggested some quite clever approaches to checking the numerical work described here. Several of the integrals were checked by his method against the method just discussed, and were found to agree to within limits of accuracy [see Rice (1973)].

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M.-J. Cross generated the stable random walks described in the text using the RAND table of random numbers to generate uniformly distributed random variables on the interval (0, 1) which were then transformed via the inverse of the distribution function into stable random variables. The details of this work as well as many more representative stable random walks are described in a forthcoming paper.

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